Varianta 095

Subiectul I

a) Punând z=0 in ecuația dreptei obținem (10,5,0). b) 1. c) $(0;\frac{5}{2})$; $(0;-\frac{5}{2})$. d) $\sqrt{3}$. e) $\sqrt{2}$. f)30.

Subiectul II 1. a) 0. b) 1. c) x=0. d) 3. e) $\frac{1}{5}$.

2. a) $f'(x)=-2\sin 2x$, $x\in \mathbf{R}$. b) $f'(\pi)=0$; c) $f(x+\pi)=\cos(2x+2\pi)=\cos 2x=f(x)$, $x\in \mathbf{R}$.

d) $\int_0^{\frac{\pi}{4}} f(x)dx=\left.\frac{\sin 2x}{2}\right|_0^{\frac{\pi}{4}}=\frac{1}{2}$. e) Deoarece $-1\le \cos 2n\le 1$, $\forall n\in \mathbf{N}$, rezultă că $\lim\limits_{n\to\infty}\frac{\cos 2n}{n}=0$.

Subiectul III

a) Fie $x_1,x_2\in$ cu **R-Q** cu $f(x_1)=f(x_2)$. Obținem $1+\frac{1}{x_1}=1+\frac{1}{x_2}$ de unde $x_1=x_2$, deci f injectivă. Fie $y\in$ **R-Q**. Din $y=1+\frac{1}{x}\Rightarrow x=\frac{1}{y-1}$ obținem că pentru orice $y\in$ **R-Q** există $x=\frac{1}{y-1}\in$ **R-Q** astfel încât $f(x)=y$,deci f surjectivă.

b) $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$, $F_6=8$, $F_7=13$ și $F_n=F_{n-1}+F_{n-2}\ \ge F_7=13$, pentru orice $n\ge 8$, deci $F_n\ne 10, \forall n\in \mathbf{N}$.($(F_n)$ strict cresc.). c) $F_8=F_7+F_6=21$, deci n=8.

d) Demonstrăm prin inducție. Pentru n = 0 este evident (0 = 0).

Dacă $F_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ și $F_{n-1}=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$ atunci

$F_n+F_{n-1}=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)=F_{n+1}$.

e) Fie $P(n): a_n=f_n(a_0)$, $n\in \mathbf{N}^*$.P(1); $a_1=f_1(a_0)$ adev.Fie P(k), $k\in \mathbf{N}^*$ adev., deci $a_k=f_k(a_0)$ și avem $a_{k+1}=f(a_k)=f(f_k(a_0))=f_{k+1}(a_0)$, deci P(k+1) adev. $\Rightarrow$ P(n) adev. pt. $\forall n\in \mathbf{N}^*$.

f) P(n): $f_n(x)=\frac{F_{n+1}x+F_n}{F_nx+F_{n-1}}, x\in \mathbf{R}\backslash\mathbf{Q},\ n\in \mathbf{N}^*$. P(1): $f(x)=\frac{F_2x+F_1}{F_1x+F_0}$ este adevărată.

Considerăm P(k) adev., deci $f_k(x)=\frac{F_{k+1}x+F_k}{F_kx+F_{k-1}}$ și avem $f_{k+1}(x)=(f\circ f_k)(x)=1+\frac{1}{f_k(x)}=1+\frac{F_kx+F_{k-1}}{F_{k+1}x+F_k}=\frac{F_{k+1}x+F_kx+F_k+F_{k-1}}{F_{k+1}x+F_k}==\frac{F_{k+2}x+F_{k+1}}{F_{k+1}x+F_k}$, deci P(k+1) adev. P(n) adev $\forall n\in \mathbf{N}^*$.

g) $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}f_n(a_0)=\lim\limits_{n\to\infty}\frac{F_{n+1}a_0+F_n}{F_na_0+F_{n-1}}=$

$$\lim_{n\to\infty}\frac{\left[\left(\frac{1+\sqrt5}{2}\right)^{n+1}-\left(\frac{1-\sqrt5}{2}\right)^{n+1}\right]a_0+\left(\frac{1+\sqrt5}{2}\right)^{n}-\left(\frac{1-\sqrt5}{2}\right)^{n}}{\left[\left(\frac{1+\sqrt5}{2}\right)^{n}-\left(\frac{1-\sqrt5}{2}\right)^{n}\right]a_0+\left(\frac{1+\sqrt5}{2}\right)^{n-1}-\left(\frac{1-\sqrt5}{2}\right)^{n-1}}=\frac{1+\sqrt5}{2}.$$

Subiectul IV

a) Avem $tg\left(\frac{\pi}{2}-x\right)=ctgx$ şi $ctg\left(\frac{\pi}{2}-x\right)=tgx$, $x\in\left(0,\frac{\pi}{2}\right)$.

b) $\lim\limits_{x\to\frac{\pi}{2}} f_n(x)=\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{1}{1+\frac{ctg^n x}{tg^n x}}\right)=1$, si analog $\lim\limits_{x\to 0} g_n(x)=1$.

c) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} f_1(x)dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{tgx}{tgx+ctgx}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin^2 xdx=\frac{1}{2}(x-\sin x\cos x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{1}{2}\left(\frac{\pi}{12}+\frac{\sqrt3-2}{4}\right)$.

d) $f_n(x)=1-\frac{1}{tg^{2n}x+1}, x\in\left(0,\frac{\pi}{2}\right), n\in\mathbf{N}.$ Fie $x_1,x_2\in\left(0,\frac{\pi}{2}\right), x_1<x_2$. Funcția tg este strict crescătoare pe $\left(0,\frac{\pi}{2}\right)$ şi obținem $f_n(x_1)<f_n(x_2)$, deci f_n este strict crescătoare. Atunci $f_n\left(\frac{\pi}{2}-x_1\right)>f_n\left(\frac{\pi}{2}-x_2\right)$, deci $g_n(x_1)>g_n(x_2)$, de unde g_n descrescătoare.

e) $I_1=\int_{\frac{\pi}{4}-a}^{\frac{\pi}{4}+a} f_n(x)dx=\int_{\frac{\pi}{4}-a}^{\frac{\pi}{4}+a} g_n(\frac{\pi}{2}-x)dx=-\int_{-\frac{\pi}{4}+a}^{-\frac{\pi}{4}-a} g_n(t)dt=\int_{\frac{\pi}{4}-a}^{\frac{\pi}{4}+a} g_n(-x)dx$.

$g_n(-x)=g_n(x), \forall x\in\left(0,\frac{\pi}{2}\right)\Rightarrow I_1=\int_{\frac{\pi}{4}-a}^{\frac{\pi}{4}+a} g_n(x)dx=I_2. I_1+I_2=\int_{\frac{\pi}{4}-a}^{\frac{\pi}{4}+a}dx=2a.$ Deci $I_1=I_2=a$.

f) $0\le\int_a^b f_n(x)dx\le\int_a^b\frac{tg^n x}{2}dx\le\frac{1}{2}\int_a^b tg^n bdx=\frac{1}{2}(b-a)(tgb)^n\Rightarrow\lim\limits_{n\to\infty}\int_a^b f_n(x)dx=0$.

g) Avem $\int_{\frac{\pi}{10}}^{\frac{\pi}{3}} f_n(x)dx=\int_{\frac{\pi}{10}}^{\frac{\pi}{6}} f_n(x)dx+\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} f_n(x)dx=\frac{\pi}{12}+\int_{\frac{\pi}{10}}^{\frac{\pi}{6}} f_n(x)dx$ şi, din f), rezultă că

$$\lim_{n\to\infty}\int_{\frac{\pi}{10}}^{\frac{\pi}{3}} f_n(x)dx=\frac{\pi}{12}.$$